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Asymptotic behaviors of radially symmetric solutions of $\Box u=|u|^{p}$ for super critical values p in high dimensions

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1. Introduction

We study asymptotic behaviors as $t\to\pm\infty$ of radially symmetric solutions of the nonlinear wave equation

(1.1) $$u_{tt}-\Delta u=F(u) \quad \text{in} \quad x\in \mathbb{R}^{n}, t\in \mathbb{R},$$

where $F(u)=|u|^{p}$ or $F(u)=|u|^{p-1}u$ with $p>1$ and $n\geqq 2$.

Let $p_{0}(n)$ be the positive root of the quadratic equation in p:

(1.2) $$\Phi(n,p)\equiv\frac{n-1}{2}p^{2}-\frac{n+1}{2}p-1=0.$$

Note that $p_{0}(n)$ is strictly decreasing with respect to n and $p_{0}(4)=2$. If $1<p<p_{0}(n)$, it is known that the Cauchy problem for (1.1) with initial data prescribed on $t=0$ does not admit global (in time) solutions, provided the initial data are chosen appropriately, even if they are sufficiently small. (See [6], [8] and [19]). The same is true for $p=p_{0}(n)$ if $n=2$ or $n=3$. (See [18]).

On the other hand, the case where $p>p_{0}(n)$ seems to be more complicated. When $2\leqq n\leqq 4$, it is known that the problem admits a global solution for small initial data. (See [7], [8] and [24]). When $n\geqq 5$, for $p\geqq(n+3)/(n-1)$ a global weak solution of the problem obtained by [13] and [20]. (See also [3], [4], [11] and [12]). Recentely, the case where p is between $p_{0}(n)$ and $(n+3)/(n-1)$ is treated by [5] and [14], independently.

Moreover, when $p>p_{0}(n)$ and either $n=2$ or $n=3$, it has been shown that the scattering operator for (1.1) exists on a dense set of a neighborhood of 0 in the energy space. (See [10],

[17] and [23]). Namely, let $u_-(x,t)$ be the solution of the homogeneous wave equation

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \tag{1.3}$$

with small initial data

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for} \quad x \in \mathbb{R}^n.$$

Then there exists a solution $u(x,t)$ of (1.1) such that $\|u(t) - u_-(t)\|_e \to 0$ as $t \to -\infty$, where

$$\|v(t)\|_e = \left\{ \int_{\mathbb{R}^n} (|\nabla v(x,t)|^2 + |v_t(x,t)|^2) dx \right\}^{1/2}, \tag{1.4}$$

and there exists another solution $u_+(x,t)$ of (1.3) such that $\|u(t) - u_+(t)\|_e \to 0$ as $t \to \infty$. The analogous results have been obtained also for the high dimensional case, provided $p > p_1(n)$, where $p_1(n)$ is the largest root of the quadratic equation in p:

$$(n^2 - n)p^2 - (n^2 + 3n - 2)p + 2 = 0.$$

(See [13], [15], [16], and [20]). However here is a gap between $p_0(n)$ and $p_1(n)$. Indeed, since the left-hand-side of the above quadratic equation is rewritten as

$$2\{n\Phi(n,p) - 2(1 + \Phi(n,p))/p\},$$

it is easy to see that $p_0(n) < p_1(n)$.

The purpose of this note is to search the asymptotic behaviors of radially symmetric solutions of (1.1), which guarantee the existence of the scattering operator, for $p > p_0(n)$ in high dimensions $n \geqq 5$.

2. Statements of main results

Throughout this section, we assume $n \geqq 5$ (unless stated otherwise). First we shall consider the Cauchy problem for the homogeneous wave equation:

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0 \quad \text{in} \quad \Omega, \tag*{(2.1)$_0$}$$

$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{for} \quad r > 0, \tag*{(2.1)$_1$}$$

where $\Omega = \{(r,t) \in \mathbb{R}^2;\ r > 0\}$ and $u(r,t)$ a real valued function. Then we have

Theorem 1. *Assume $f \in C^2([0,\infty))$ and $g \in C^1([0,\infty))$ satisfy*

$$|f(r)|\langle r\rangle^{-1} + \sum_{j=0}^{1}(|f^{(j+1)}| + |g^{(j)}(r)|) \leqq \varepsilon\langle r\rangle^{-\kappa-(n+1)/2} \quad \text{for} \quad r > 0, \tag{2.2}$$

where ε and κ are positive numbers and $\langle r\rangle = \sqrt{1+r^2}$. Here if n is even number, we further assume $\kappa < (n-1)/2$. Then (2.1) admits uniquely a weak solution $u(r,t) \in C^1(\Omega)$ such that for $(r,t) \in \Omega$ and $|\alpha| \leqq 1$ we have

$$|D^\alpha_{r,t}u(r,t)| \leqq C\varepsilon r^{1-m-|\alpha|}\langle r\rangle^{-1+|\alpha|}\Psi(r,|t|), \tag{2.3}$$

where we have set $m = [(n-2)/2]$ and

$$\Psi(r,t) = \langle r+|t|\rangle^{-\chi(n)}\langle r-t\rangle^{-\kappa}$$

with

$$\chi(n) = \begin{cases} 1/2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

and C is a constant depending only on m and κ.

Next we shall consider the nonlinear wave equation

$$u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F(u) \quad \text{in} \quad \Omega, \tag{2.4}$$

where $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$. Here we assume

$$p_0(n) < p < (n+3)/(n-1). \tag{2.5}$$

We shall introduce a function space X, in which we will look for solutions of (2.4), defined by

$$X = \{u(r,t) \in C^0(\Omega) : \; D_r u(r,t) \in C^0(\Omega), \; \|u\| < \infty\},$$

and

$$\|u\| = \sup_{(r,t)\in\Omega}\{(|u(r,t)|r^{m-1}\langle r\rangle + |D_r u(r,t)|r^m)\Psi^{-1}(r,|t|)\},$$

where Ψ is the same function as in (2.3). As for the parameter κ, we assume

$$\frac{1}{2} < \kappa \quad \text{and} \quad \frac{p+1}{p-1} - \frac{n+1}{2} < \kappa \leqq q, \tag{2.6}$$

where we have set

$$q = (1 + \Phi(n,p))/p = \frac{n-1}{2}p - \frac{n+1}{2}$$

with $\Phi(n,p)$ in (1.2). Note that there exist really numbers κ satisfying (2.6) for $p > p_0(n)$, because

$$\Phi(n,p) = (p-1)\{q - (\frac{p+1}{p-1} - \frac{n+1}{2})\} > 0 \quad \text{for} \quad p > p_0(n).$$

We are now in a position to state the main theorem in this note. Let $u_-(r,t)$ be the solution of (2.1) which is obtained in Theorem 1. Note that $u_- \in X$ and

$$\|u_-\| \leqq C\varepsilon \quad \text{for any} \quad \varepsilon > 0. \tag{2.7}$$

Then we have

Theorem 2. (Main theorem). *Assume conditions (2.2), (2.5) and (2.6) hold. Then there is positive constant ε_0 (depending only on p, n and κ) such that, if $0 < \varepsilon \leqq \varepsilon_0$, there exists uniquely a weak solution $u(r,t)$ of the nonlinear wave equation (2.4) such that $u \in C^1(\Omega) \cap X$,*

$$\|u\| \leqq 2\|u_-\| \tag{2.7}$$

and for $(r,t) \in \Omega$ and $|\alpha| \leqq 1$ we have

$$|D^\alpha_{r,t}(u(r,t) - u_-(r,t))| \leqq C\|u\|^p r^{1-m-|\alpha|}\langle r\rangle^{-1+|\alpha|}\Psi(r,t) \tag*{(2.8)$_-$}$$

and

$$\|u(t) - u_-(t)\|_e \leqq C\|u\|^p \langle t\rangle^{-\theta} \quad \textit{if} \quad t \leqq 0, \tag*{(2.9)$_-$}$$

where $\|\cdot\|$ is defined by (1.4) and we have set

$$\theta = \min\{q, \chi(n)p + p\kappa - 1\},$$

and C is a constant depending only on p, n and κ.

Moreover there exists uniquely a weak solution $u_+(r,t)$ of $(2.1)_0$ which belongs to $C^1(\Omega) \cap X$, such that for $(r,t) \in \Omega$ and $|\alpha| \leqq 1$ we have

$$|D^\alpha_{r,t}(u(r,t) - u_+(r,t))| \leqq C\|u\|^p r^{1-m-|\alpha|}\langle r\rangle^{-1+|\alpha|}\Psi(r,-t) \tag*{(2.8)$_+$}$$

and

$$\|u(t) - u_+(t)\|_e \leqq C\|u\|^p \langle t\rangle^{-\theta} \quad \text{if} \quad t \geqq 0. \tag{2.9$_+$}$$

Remarks. 1) If n is odd, in Theorems 1 and 2, one can replace $u \in C^1(\Omega)$ by $u \in C^2(\Omega)$. Moreover in (2.6) we can replace $\kappa > 1/2$ by $\kappa > 0$. In this case, we interpret $(2.9)_\pm$ as follows. When $\kappa > 1/2p$, $(2.9)_\pm$ is still valid. When $0 < \kappa \leqq 1/2p$, it holds with $\theta = \kappa$. (See [9]).

2) For $n \geqq 2$, consider the following Cauchy problem

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{n-1}{r} u_r = F(u) & \text{in} \quad r > 0,\ t > 0, \\ u(r,0) = 0, \quad u_t(r,0) = g(r) & \text{for} \quad r > 0. \end{cases} \tag{2.10}$$

It is known that, if $g(r) \geqq Mr^{-\mu}$ for $r \geqq 1$ with some positive constants M, μ and $\mu < (p+1)/(p-1)$, then (2.10) does not admit global solutions. (See [1], [2], [21] and [22]). Therefore condition (2.6) is partially necessary to obtain Theorem 2.

3) One can also show that the Cauchy problem for the nonlinear wave equation (2.4) admits a unique global solution, provided the hypotheses of Theorems 1 and 2 are fulfilled.

In the proof of Theorem 1, the following lemma plays a key role. Moreover Theorem 2 is obtained by considering the assosiated integral equation with the differntial equation (2.4). So the lemma below is very essential in our work.

Lemma 3. *Let $g \in C^0((0,\infty))$ and*

$$g(r) = O(r^{-m-1}) \quad \text{as } r \downarrow 0.$$

For $r > 0$ and $t \geqq 0$ we define a function $\Theta(g)$ as follows.

(1) *n is odd : $n = 2m+3$ $(m = 1, 2, \cdots)$.*

$$\Theta(g)(r,t) = \int_{|t-r|}^{t+r} g(\lambda) K(\lambda, r, t)\, d\lambda,$$

where we have set

$$K(\lambda, r, t) = r^{2-n} \lambda^{2m+1} H_m(\lambda, r, t),$$
$$H_m(\lambda, r, t) = \Big(\frac{\partial}{\partial\lambda}\frac{-1}{2\lambda}\Big)^m (r^2 - (\lambda - t)^2)^{(n-3)/2}.$$

(2) *n is even* : $n = 2m+2$ $(m = 1, 2, \cdots)$.

$$\Theta(g)(r,t) = \int_{|t-r|}^{t+r} g(\lambda)K_1(\lambda,r,t)d\lambda + \int_0^{\max(t-r,0)} g(\lambda)K_2(\lambda,r,t)d\lambda,$$

where we have set

$$K_1(\lambda,r,t) = r^{2-n}\lambda^{2m+1}\int_{\lambda}^{t+r}\frac{H_m(\rho,r,t)}{\sqrt{\rho^2-\lambda^2}}d\rho,$$
$$K_2(\lambda,r,t) = r^{2-n}\lambda^{2m+1}\int_{t-r}^{t+r}\frac{H_m(\rho,r,t)}{\sqrt{\rho^2-\lambda^2}}d\rho,$$

and

$$H_m(\rho,r,t) = (\frac{\partial}{\partial\rho}\frac{-1}{2\rho})^m(r^2-(\rho-t)^2)^{(n-3)/2}.$$

And we extend $\Theta(g)(r,t)$ as an odd function with respect to t. Then $\Theta(g) \in C^0(\Omega)$ and for each bounded subset $B \subset \Omega$ we have

$$|\Theta(g)(r,t)| \leq C_B\ r^{-m} \quad \text{for} \quad (r,t) \in B.$$

Moreover, if we set $u(x,t) = \Theta(g)(|x|,t)$, then $u(\cdot,t) \in C^0(\mathbb{R}; L^2_{loc}(\mathbb{R}^n))$ and u is a weak solution of the Cauchy problem

$$\begin{cases} u_{tt} - \triangle u = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = 0, \quad u_t(x,0) = c_n g(|x|) & \text{for} \quad x \in \mathbb{R}^n \end{cases}$$

in the sense of disribution, where

$$c_n = \begin{cases} 2\ \Gamma(\frac{n-1}{2}) & \text{if } n \text{ is odd}, \\ \sqrt{\pi}\ \Gamma(\frac{n-1}{2}) & \text{if } n \text{ is even}. \end{cases}$$

Furthermore, if $g \in C^1((0,\infty))$ and for $j = 0, 1$

$$g^{(j)}(r) = O(r^{-m-j}) \quad \text{as } r \downarrow 0,$$

then $\Theta(g) \in C^1(\Omega)$ and for each bounded subset $B \subset \Omega$ we have

$$|D^{\alpha}_{r,t}\Theta(g)(r,t)| \leq C_B\ r^{1-m-|\alpha|} \quad \text{for} \quad (r,t) \in B \text{ and } |\alpha| \leqq 1.$$

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